Brownian motion and Stochastic Calculus
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Assignment 9

Exercise 1

Let $W = (W_t)_{t\geq 0}$ be a 1-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion.

1) Prove that for every polynomial p on $\mathbb{R}$, the stochastic integral $\int_0^\cdot p(W_s)\mathrm{d}W_s$ is well defined. Moreover, show that it is also an $(\mathbb{F},\mathbb{P})$-martingale.

2) Show that the process $X = (X_t)_{t\geq 0}$ given by $X_t := \mathrm{e}^{\frac{1}{2}t}\cos(W_t)$, $t \geq 0$ is an $(\mathbb{F},\mathbb{P})$-martingale.

3) Let W' be another $(\mathbb{F},\mathbb{P})$–Brownian motion independent of W and ρ be an $\mathbb{F}$-adapted, measurable, process satisfying $|\rho| \leq 1$. Prove that the process B given by

$$B_t = \int_0^t \rho_s \mathrm{d}W_s + \int_0^t \sqrt{1-\rho_s^2}\,\mathrm{d}W'_s$$

is an $(\mathbb{F},\mathbb{P})$–Brownian motion. Moreover, compute $[B,W]$.

Exercise 2

For any $M \in \mathcal{M}_{c,\mathrm{loc}}(\mathbb{R},\mathbb{F},\mathbb{P})$, define $M^\star_t := \sup_{0\leq s\leq t}|M_s|$, for $t \geq 0$. Prove that for any $t \geq 0$ and positive C, K, we have

$$\mathbb{P}\big[M^\star_t > C\big] \leq \frac{4K}{C^2} + \mathbb{P}\big[[M]_t > K\big].$$

Exercise 3

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Consider the SDE

$$\mathrm{d}X_t = \bigg(\sqrt{1+X_t^2} + \frac{1}{2}X_t\bigg)\mathrm{d}t + \sqrt{1+X_t^2}\mathrm{d}B_t,\ X_0 = x \in \mathbb{R}. \tag{0.1}$$

1) Show that for any $x \in \mathbb{R}$, the SDE defined in (0.1) has a unique strong solution.

2) Show that $(X_t)_{t\geq 0}$ defined by $X_t := \sinh\big(\operatorname{arcsinh} x + t + B_t\big)$ is the unique solution of (0.1).

Exercise 4

Let B be a Brownian motion defined on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfying the usual conditions, and let us fix three constants $(a,b,\sigma) \in (0,+\infty)^3$, and an initial value $r_0 \in \mathbb{R}$. An Ornstein–Uhlenbeck process r satisfies the following SDE

$$r_t = r_0 + \int_0^t (a - br_s)\mathrm{d}s + \sigma B_t,\ t \geq 0.$$

1) Show that

$$r_s = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b} + \int_t^s \mathrm{e}^{-b(s-u)}\sigma \mathrm{d}B_u,\ 0 \leq t \leq s.$$

2) Deduce that the $\mathbb{P}$-distribution of r_s knowing $\mathcal{F}_t$ is Gaussian with mean

$$m(t,s) := \mathbb{E}^{\mathbb{P}}\left[r_s\middle|\mathcal{F}_t^{B,\mathbb{P}}\right] = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b},$$

and variance

$$v(t,s) := \mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[r_s\middle|\mathcal{F}_t^{B,\mathbb{P}}\right] = \sigma^2\int_t^s \mathrm{e}^{-2b(s-u)}du = \frac{\sigma^2}{2b}\left(1-\mathrm{e}^{-2b(s-t)}\right).$$

3) Prove the following stochastic Fubini theorem.

Lemma 0.1. *Let b and σ be two $\mathbb{R}$-valued measurable and $\mathbb{F}$-adapted processes such that for any $t \geq 0$*

$$\int_0^t \left(|b_s| + |\sigma_s|^2\right)\mathrm{d}s < +\infty.$$

We have for any $t \geq 0$

$$\int_0^t b_s\left(\int_0^s \sigma_u \mathrm{d}B_u\right)\mathrm{d}s = \left(\int_0^t \sigma_u \mathrm{d}B_u\right)\left(\int_0^t b_s\mathrm{d}s\right) - \int_0^t \sigma_u\left(\int_0^u b_s\mathrm{d}s\right)\mathrm{d}B_u.$$

Deduce from this that

$$\int_t^s r_u\mathrm{d}u = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t) + \sigma\int_t^s \frac{1-\mathrm{e}^{-b(s-u)}}{b}\mathrm{d}B_u,$$

and that the distribution of $\int_t^s r_u\mathrm{d}u$, conditionally on $\mathcal{F}_t$, is Gaussian with

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^s r_u\mathrm{d}u\middle|\mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t),$$

and

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\int_t^s r_u\mathrm{d}u\middle|\mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{\sigma^2}{b^2}\left(s-t-\frac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right).$$

4) Finally, prove that the joint distribution, knowing $\mathcal{F}_t$, of the vector $(r_s, \int_t^s r_u\mathrm{d}u)$ is still Gaussian with mean given by the vector

$$\begin{pmatrix} \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b} \\ \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t) \end{pmatrix},$$

and covariance matrix

$$\begin{pmatrix} \frac{\sigma^2}{2b}\left(1-\mathrm{e}^{-2b(s-t)}\right) & \sigma^2\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \frac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) \\ \sigma^2\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \frac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) & \frac{\sigma^2}{b^2}\left(s-t-\frac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right) \end{pmatrix}.$$

Exercise 5

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. Assume that the filtration $\mathbb{F}$ is generated by the Brownian motion W. Consider the Tanaka SDE

$$\mathrm{d}X_t = \mathrm{sgn}(X_t)\mathrm{d}W_t,\ X_0 = 0,$$

where $\mathrm{sgn}(x)$ denotes the sign function, i.e., $\mathrm{sgn}(x) := 1$ if $x > 0$, and $\mathrm{sgn}(x) = -1$ if $x \leq 0$.

1) Show that the Tanaka SDE has no strong solution.

Hint:

- Assume there exists a strong solution and derive a contradiction.
- You can use the following result (Tanaka's formula): let X be a continuous semimartingale. There exists a continuous, non-decreasing adapted process $(L_t)_{t\geq 0}$ such that

$$|X_t| - |X_0| = \int_0^t sgn(X_s)dX_s + L_t,\ t \geq 0.$$

  Moreover, it can be shown that L is $\mathbb{F}^{|X|}$-adapted.

2) Show that the SDE admits a weak solution.